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1999 J. Phys. A: Math. Gen. 32 5087

(http://iopscience.iop.org/0305-4470/32/27/308)

J. Phys. A: Math. Gen. **32** (1999) 5087–5096. Printed in the UK PII: S0305-4470(99)01045-8

The gauge equivalence of the NLS and the Schrödinger flow of maps in 2 + 1 dimensions

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Received 19 January 1999

Abstract. The gauge equivalence of the analogues of the nonlinear Schrödinger equation NLS^- and those of the Schrödinger flow of maps into H^2 (the Minkowski HF model) in $2+1$ dimensions are proved, respectively. Combining these with the already-known results, we obtain a complete understanding of the gauge equivalence of the analogous models of the nonlinear Schrödinger equation (for $\kappa = 1$ or -1) and those of the Heisenberg ferromagnet model (for Euclidean or Minkowski) in $2+1$ dimensions.

1. Introduction

The nonlinear Schrödinger equation (NLS)

$$\mathrm{i}\psi_t + \psi_{xx} + 2\kappa|\psi|^2\psi = 0 \tag{1}$$

where the subscripts denote partial derivatives and κ is a constant, arises in physics from varied backgrounds, such as in plasma physics and nonlinear optics, and provides a fairly universal model of a nonlinear equation. Without loss of generality, we will denote by NLS^+ and NLS^- the nonlinear Schrödinger equation (1) with $\kappa = 1$ and $\kappa = -1$, respectively. It is well known (see [1,2]) that there is a gauge equivalence between the NLS^+ and the Heisenberg ferromagnet model (the HF model): $u_t = u \times u_{xx}$, where $u = (u_1, u_2, u_3)$ is the coordinates of a point on the unit sphere in R^3, which is an important equation in condensed-matter physics. Although the dynamical properties of the NLS^+ and NLS^- are very different (for example, the NLS^+ has light soliton solutions and the NLS^- has no light soliton solutions, but rather has dark soliton solutions), we have known that there is a complete unified geometric interpretation of the NLS (1) for $\kappa = 1, 0, -1$ from the recent work in [3]. That is: they are exactly gauge equivalent to the Schrödinger flow of maps from R^1 into the Euclidean 2-space S^2 (with Gauss curvature 1) [2], the complex plane C (with Gauss curvature 0) and the hyperbolic 2-space H^2 (with Gauss curvature -1) [3], respectively. Because the continuous Heisenberg ferromagnet equation (the HF model) is simply the Schrödinger flow of maps from R^1 into the Euclidean 2-sphere S^2 and the hyperbolic 2-space H^2 used in [3] is a similar unit sphere in Minkowski 3-space R^{2+1} (see [3] for details), we regard the Schrödinger flow of maps from R^1 into

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the hyperbolic 2-space H^2 as the Minkowski continuous Heisenberg ferromagnet model (the M-HF model) in this paper.

Much effort has been devoted to the study of (2 + 1)-dimensional integrable systems [4–6, 9, 10, 13–15]. Here we have the following interesting phenomenon: for a (1 + 1)-dimensional integrable soliton equation, there exist usually two different ways in obtaining its (2 + 1)-dimensional integrable generalizations. One standard way in constructing the (2 + 1)-dimensional models is to start with the linear problem for a (1 + 1)-dimensional model, and then replace the spectral parameter by a differential operator. For the NLS this process yields the Davey–Stewartson equation (DS_{II}) and it takes the following form:

$$\mathrm{i}q_t - q_{xx} + q_{yy} + 2\phi q = 0 \tag{2}$$

$$\phi_{xx} + \phi_{yy} + \kappa(q\overline{q})_{xx} - \kappa(q\overline{q})_{yy} = 0 \tag{3}$$

where $q = q(t, x, y)$, etc, and the bar denotes the complex conjugate. We denote by DS^+_{II} and DS^-_{II} the DS_{II} with $\kappa = 1$ and $\kappa = -1$, repectively. Another method for obtaining a (2 + 1)-dimensional integrable model from a (1 + 1)-dimensional one, as done by Fordy *et al* [7, 8], is Lie algebraic. For the NLS the latter process leads to the so-called (2 + 1)-dimensional NLS as follows:

$$\mathrm{i}\psi_t + \psi_{xy} + 2\kappa\psi\partial_x^{-1}\partial_y|\psi|^2 = 0 \tag{4}$$

and we similarly denote by (2 + 1) NLS^+ and (2 + 1) NLS^- the (2 + 1)-dimensional nonlinear Schrödinger equation (4) with $\kappa = 1$ and $\kappa = -1$, respectively.

The gauge equivalent structure of the (1 + 1) NLS is now very well understood (see [2, 3]). However, generally speaking, in (2 + 1)-dimensional integrable systems we have a number of remarkable properties, which may not exist in their (1 + 1)-dimensional countparts (for example, see [10] for some comments). So an interesting question that naturally arises is whether the analogues of the NLS in 2 + 1 dimensions with such an important gauge equivalent structure exist. In 1990, Cheng *et al* proved [9] that the DS^+_{II} is gauge equivalent to the following Ishimori equation [6] which is obtained in the same way as the DS_{II} is from the HF model:

$$\boldsymbol{S}_t + \boldsymbol{S} \times (\boldsymbol{S}_{xx} - \boldsymbol{S}_{yy}) + \phi_x\boldsymbol{S}_y + \phi_y\boldsymbol{S}_x = 0 \tag{5}$$

$$\phi_{xx} + \phi_{yy} - 2\boldsymbol{S}(\boldsymbol{S}_x \times \boldsymbol{S}_y) = 0 \tag{6}$$

where $\boldsymbol{S} = (s_1(t, x, y), s_2(t, x, y), s_3(t, x, y)) \in R^3$ with $|\boldsymbol{S}|^2 = 1$. In 1998, Myrzakulov *et al* demonstrated in [10, 15] that the (2 + 1) NLS^+ is gauge equivalent to the following HF model in 2 + 1 dimensions obtained in the same way as the (2 + 1) NLS^+:

$$\boldsymbol{S}_t = (\boldsymbol{S} \times \boldsymbol{S}_y + u\boldsymbol{S})_x \tag{7}$$

$$u_x = -\boldsymbol{S} \cdot (\boldsymbol{S}_x \times \boldsymbol{S}_y) \tag{8}$$

where $\boldsymbol{S} = (s_1, s_2, s_3) \in R^3$ with $|\boldsymbol{S}|^2 = 1$ and $\times$ denotes the cross product.

The purpose of this paper is to show that the analogues of the NLS^- in 2 + 1 dimensions are respectively gauge equivalent to the analogues of the M-HF model in 2 + 1 dimensions. Namely, we prove firstly that the (2 + 1) NLS^- is gauge equivalent to the following (2 + 1) M-HF model obtained in a similar way as (7), (8) from the system (i.e. [3, equation (5)]) of the Schrödinger flow of maps into $H^2 \hookrightarrow R^{2+1}$ (the M-HF model):

$$\boldsymbol{S}_t = (\boldsymbol{S} \dot{\times} \boldsymbol{S}_y + 2\mathrm{i}u\boldsymbol{S})_x \tag{9}$$

$$u_x = -\frac{1}{2\mathrm{i}}\boldsymbol{S} \cdot (\boldsymbol{S}_x \dot{\times} \boldsymbol{S}_y) \tag{10}$$

where $\boldsymbol{S} = (s_1(t,x,y), s_2(t,x,y), s_3(t,x,y)) \in R^{2+1}$ with $|\boldsymbol{S}|^2 = s_1^2 + s_2^2 - s_3^2 = -1$ and $s_3 > 0$, and $\dot{\times}$ denotes the pseudo-cross product, i.e. for two vectors $\boldsymbol{a}, \boldsymbol{b} \in R^{2+1}$

$$\boldsymbol{a} \dot{\times} \boldsymbol{b} = (a_2 b_3 - a_3 b_2, a_3 b_1 - a_1 b_3, -(a_1 b_2 - a_2 b_1)).$$

Secondly, we show that the $\mathrm{DS}_{\mathrm{II}}^-$ is also gauge equivalent to the following $(2+1)$-dimensional integrable system obtained in a similar way to (5), (6) from the M-HF model, which we call the Minkowski Ishimori equation (Minkowski IE):

$$\boldsymbol{S}_t + \boldsymbol{S} \dot{\times} (\boldsymbol{S}_{xx} - \boldsymbol{S}_{yy}) + \phi_x \boldsymbol{S}_y + \phi_y \boldsymbol{S}_x = 0 \tag{11}$$

$$\phi_{xx} + \phi_{yy} + 2\boldsymbol{S}(\boldsymbol{S}_x \dot{\times} \boldsymbol{S}_y) = 0 \tag{12}$$

where $\boldsymbol{S} = (s_1(t,x,y), s_2(t,x,y), s_3(t,x,y)) \in R^{2+1}$ with $|\boldsymbol{S}|^2 = -1$ and $s_3 > 0$. These are dual interpretations of the gauge equivalence between the analogous models of the NLS^+ eqaution and those of the HF model in 2 + 1 dimensions. An effective method applied here is a different choice of Lax pair for the M-HF model and the Minkowski Ishimori equation according to the dynamical behavior of auxiliary linear problem of the (2 + 1) NLS^- and the $\mathrm{DS}_{\mathrm{II}}^-$. And we use it to relate the M-HF model or the Minkowski Ishimori equation to the (2 + 1) NLS^- or the $\mathrm{DS}_{\mathrm{II}}^-$ in a natural way by gauge transformation. Our arguments will depend completely on the dynamical properties of the (2 + 1) NLS^- or $\mathrm{DS}_{\mathrm{II}}^-$.

2. Gauge equivalence between the (2 + 1) NLS^- and the M-HF model

In this section, we show that the (2+1) NLS^- is gauge equivalent to the M-HF model. Because the NLS^- has no light-soliton solutions and neither does the (2+1) NLS^-, we put $\phi = \psi \mathrm{e}^{-\mathrm{i}\rho^2 t}$, where ρ is a positive real constant, and get an equivalent equation for ϕ:

$$\mathrm{i}\phi_t + \phi_{xx} - 2(\partial_x^{-1}\partial_y |\phi|^2 - \rho^2)\phi = 0. \tag{13}$$

As pointed out in [1], in order to solve (13) we need to add the following finite density boundary condition at infinity:

$$\begin{aligned} &\phi \to \rho && \text{as } x \to +\infty \\ &\phi \to \rho \mathrm{e}^{\mathrm{i}2\beta} && \text{as } x \to -\infty \end{aligned} \tag{14}$$

where β is a real constant. According to the process of Fordy *et al* (see [7, 8]), and using the Lax pair (12) for the (1 + 1) NLS^- given in [3], we can easily see that (13) (with the boundary condition (14)) permits the following Lax pair:

$$\begin{aligned} F'_x(t,x,y,\lambda) &= (\lambda\sigma_3 + U)F'(t,x,y,\lambda) \\ F'_t(t,x,y,\lambda) &= -2\mathrm{i}\lambda F'_y(t,x,y,\lambda) + \mathrm{i}\{\partial_x^{-1}\partial_y U^2 - \rho^2 + U_y\}\sigma_3 F'(t,x,y,\lambda) \end{aligned} \tag{15}$$

where

$$U = \begin{pmatrix} 0 & \phi(t,x,y) \\ \overline{\phi}(t,x,y) & 0 \end{pmatrix} \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

It is obvious that the Lax pair (15) has different dynamical behaviour for the spectral parameter λ in the range of $|\lambda| > \rho$ or $|\lambda| < \rho$. This causes some technical difficulties in characterizing the dynamical properties of (2 + 1) NLS^-. However, it is this dynamical properties of the (2 + 1) NLS^- that allows us to construct a gauge transformation to the (2 + 1) M-HF model.

For the (2 + 1) M-HF model (9), (10), we convert it into the matrix form. As done in [3], let us put

$$p_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad p_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \qquad p_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and set

$$\widetilde{S} = s_1 p_1 + s_2 \mathrm{i} p_2 + s_3 \mathrm{i} p_3 \tag{16}$$

for $\boldsymbol{S}(t,x,y) \in H^2$. Obviously, $\widetilde{S}^2 = -I$, $\operatorname{tr}\widetilde{S} = 0$, the diagonal of $\widetilde{S}$ is a real matrix and the off-diagonal of $\widetilde{S}$ is a purely imaginary matrix. Using the commutative relations: $[p_1, p_2] = -2p_3$, $[p_1, p_3] = -2p_2$ and $[p_2, p_3] = -2p_1$, we obtain, by a direct computation, the result that (9), (10) can be rewritten as

$$\widetilde{S}_t = \left(\frac{1}{2}\left[\widetilde{S}, \widetilde{S}_y\right] + 2\mathrm{i}u\widetilde{S}\right)_x \qquad u_x = -\frac{1}{4\mathrm{i}}\operatorname{tr}(\widetilde{S}\widetilde{S}_x\widetilde{S}_y). \tag{17}$$

Similarly, we get the result that (17) permits a Lax pair as follows:

$$\begin{aligned} &\widetilde{F}_x(t,x,y,\lambda) = \mathrm{i}\lambda\widetilde{S}\widetilde{F}(t,x,y,\lambda) \\ &\widetilde{F}_t(t,x,y,\lambda) = -2\mathrm{i}\lambda\widetilde{F}_y(t,x,y,\lambda) + \mathrm{i}\lambda[\widetilde{S}\widetilde{S}_y + 2\mathrm{i}u\widetilde{S}]\widetilde{F}(t,x,y,\lambda) \end{aligned} \tag{18}$$

by using the Lax pair (9) of the $(1+1)$-dimensional M-HF model in [3], where λ is a spectral parameter.

Firstly, suppose that $\phi(t,x,y)$ be a solution to the $(2+1)$ NLS^- (13) with the boundary condition (14). The corresponding solution to the Lax pair (15) is denoted by $F'(t,x,y,\lambda)$. Consider the following gauge transformation:

$$F'(t,x,y,\lambda) = G(t,x,y)F(t,x,y,\lambda) \tag{19}$$

where $G(t,x,y)$ will be determined later. We hope that the above $F(t,x,y,\lambda)$ is a solution to Lax pair (18) of (12). In order to do this, we put $\partial_x F = LF$ and apply the first equation of the Lax pair (15): then we have

$$\lambda\sigma_3 + U = G(t,x,y)L(t,x,y,\lambda)G^{-1}(t,x,y) + G_x(t,x,y)G(t,x,y) \tag{20}$$

from (19). Substituting $L(t,x,y,\lambda) = \mathrm{i}\lambda\widetilde{S}(t,x,y)$ into (20) and comparing the coefficients of λ^j (for $j = 1, 0$) in equation (20), we obtain

$$\sigma_3 = G(t,x,y)\mathrm{i}\widetilde{S}G^{-1}(t,x,y) \qquad \text{i.e. } \widetilde{S} = -G^{-1}(t,x,y)\mathrm{i}\sigma_3 G(t,x,y) \tag{21}$$

and

$$U(t,x,y) = G_x(t,x,y)G^{-1}(t,x,y) \qquad \text{i.e. } \partial_x G(t,x,y) = U(t,x,y)G(t,x,y). \tag{22}$$

As remarked in [3], equation (22) implies that $G(t,x,y)$ satisfies the first Lax equation of (15) for $\lambda = 0$. One may check directly that solutions to such an equation are of the form:

$$G(t,x,y) = \begin{pmatrix} f(t,x,y) & g(t,x,y) \\ \bar{f}(t,x,y) & -\bar{g}(t,x,y) \end{pmatrix} \tag{23}$$

and hence, in this way, the $\widetilde{S}$ being defined by (21) coincides with the restrictions on $\widetilde{S}$ in (16).

Using the second Lax equation for F', we have

$$\partial_t F(t,x,y,\lambda) = -2\mathrm{i}\lambda F_y(t,x,y,\lambda) + M(t,x,y,\lambda)F(t,x,y,\lambda)$$

with

$$\mathrm{i}\{\partial_x^{-1}\partial_y U^2 - \rho^2 + U_y\}\sigma_3 = G_t G + 2\mathrm{i}\lambda G_y G^{-1} + GMG^{-1}. \tag{24}$$

We now show that the above M exactly equals $\mathrm{i}\lambda[\widetilde{S}\widetilde{S}_y + 2\mathrm{i}u\widetilde{S}]$, i.e. $F(t,x,y,\lambda)$ satisfies the second Lax equation in (18) with the choice of $G(t,x,y) = F'(t,x,y,0)$. In fact, on substituting $M = \mathrm{i}\lambda[\widetilde{S}\widetilde{S}_y + 2\mathrm{i}u\widetilde{S}]$ into (24), the constant term leads to

$$\mathrm{i}\{\partial_x^{-1}\partial_y U^2 + U_x\}\sigma_3 = G_t G^{-1} \qquad \text{or} \qquad \partial_t G = \mathrm{i}\{\partial_x^{-1}\partial_y U^2 - \rho^2 + U_x\}\sigma_3 G \tag{25}$$

which is satisfied by the chosen $G(t, x, y)$. So what remains for us to show is that the coefficient of λ, $\mathrm{i}2G_yG^{-1} + \mathrm{i}G[\widetilde{S}\widetilde{S}_y + 2\mathrm{i}u\widetilde{S}]G^{-1}$, on the right-hand side of (24) vanishes, i.e.

$$\widetilde{S}\widetilde{S}_y + 2\mathrm{i}u\widetilde{S} = -2G^{-1}G_y. \tag{26}$$

Indeed, from the first Lax equations of G we have

$$G_y = VG \tag{27}$$

for some matrix V satisfying the integrablity equation $U_y - V_x + [U, V] = 0$. It can be straightfowardly verified that the general form of V is

$$V = \begin{pmatrix} \mathrm{i}\alpha & \beta \\ \overline{\beta} & -\mathrm{i}\alpha \end{pmatrix}$$

for some real function α and complex function β. Now let us set $u = -\mathrm{i}\alpha$, then

$$\begin{aligned}\widetilde{S}\widetilde{S}_y + 2\mathrm{i}u\widetilde{S} &= -G^{-1}i\sigma_3G_yG^{-1}\mathrm{i}\sigma_3G - G^{-1}G_y - 2\mathrm{i}uG^{-1}\mathrm{i}\sigma_3G \\ &= G^{-1}\sigma_3V\sigma_3G - G^{-1}VG + 2uG^{-1}\sigma_3G \\ &= -2G^{-1}VG = -2G^{-1}G_y\end{aligned}$$

where we have used the facts that $\sigma_3V^{(\mathrm{diag})} = V^{(\mathrm{diag})}\sigma_3$ and $\sigma_3V^{(\text{off-diag})} = -V^{(\text{off-diag})}\sigma_3$. Thus we arrive at the desired identity (26). This proves that the matrix $\widetilde{S}$ and the function u constructed from a solution $\phi(t, x, y)$ to the (2 + 1) NLS$^-$ satisfy the system of the (2 + 1)-dimensional M-HF model (17).

Next we shall prove that the above transformation from the (2 + 1) NLS$^-$ to the (2 + 1) M-HF model (17) is in fact reversible. Suppose a matrix $\widetilde{S}$ of the form (16) and a function u satisfy equation (17). As shown in [3], we may choose a matrix $G(t, x, y)$ with the following form:

$$G(t, x, y) = \begin{pmatrix} f(t, x, y) & g(t, x, y) \\ \overline{f}(t, x, y) & -\overline{g}(t, x, y) \end{pmatrix}$$

such that $\det G = 1$, $\sigma_3 = G\mathrm{i}\widetilde{S}G^{-1}$ and

$$G_x(t, x, y)G(t, x, y)^{-1} = \begin{pmatrix} 0 & \phi(t, x, y) \\ \overline{\phi}(t, x, y) & 0 \end{pmatrix} = U(t, x, y) \tag{28}$$

for some complex function $\phi(t, x, y)$.

Because of (28), we have

$$G_y = VG$$

with

$$V = \begin{pmatrix} \mathrm{i}\alpha & \beta \\ \overline{\beta} & -\mathrm{i}\alpha \end{pmatrix}$$

satisfying the integrablity equation $U_y - V_x + [U, V] = 0$, where α is a real function and β a complex function. It is a direct calculation that the second equation of (17) may be re-expressed as follows:

$$u_x = \frac{1}{4\mathrm{i}}\,\mathrm{tr}\big(\widetilde{S}\widetilde{S}_x\widetilde{S}_y\big) = -\overline{\beta}\phi + \beta\overline{\phi}.$$

Notice that $-\mathrm{i}\alpha$ also satisfies the above equation, i.e. $-\mathrm{i}\alpha_x = -\overline{\beta}\phi + \beta\overline{\phi}$, from the diagonal part of the equation $U_y - V_x + [U, V] = 0$. Hence we obtain $u = -\mathrm{i}\alpha + \mathrm{i}c$ for some function c which will be determined later. From this we obtain

$$\widetilde{S}\widetilde{S}_y + 2\mathrm{i}u\widetilde{S} = -2G^{-1}G_y + 2\mathrm{i}cG^{-1}\sigma_3G \tag{29}$$

in a similarly way as for (26). Now, put

$$L^G(\lambda) = G_x G^{-1} + G\widetilde{L}(\lambda)G^{-1} = \lambda\sigma_3 + U \tag{30}$$

$$M^G(\lambda) = G_t G^{-1} + 2\mathrm{i}\lambda G_y G^{-1} + G\widetilde{M}(\lambda)G^{-1} = G_t G^{-1} + \lambda 2\mathrm{i}cG^{-1}G \tag{31}$$

where $\widetilde{L}(\lambda) = \mathrm{i}\lambda\widetilde{S}$, $\widetilde{M}(\lambda) = \mathrm{i}\lambda[\widetilde{S}\widetilde{S}_y + 2\mathrm{i}u\widetilde{S}]$ are the coefficient matrices in (18) and, in the second identity of (31), we have used the identity (29) . Since $\widetilde{L}$ and $\widetilde{M}$ satisfy the integrability condition of (18), we have

$$\frac{\partial L^G}{\partial t} - \frac{\partial M^G}{\partial x} + [L^G, M^G] + 2\mathrm{i}\lambda L^G_y = 0. \tag{32}$$

The vanishing of the coefficient of λ^2 in (32) implies $c \equiv 0$ and the vanishing of the coefficient of λ and the diagonal part of the constant term in (32) lead to

$$G_t G^{-1} = \mathrm{i}\{\partial_x^{-1}\partial_y U^2 + U_y\}\sigma_3 + \mathrm{i}\tau(t)\sigma_3 \tag{33}$$

for some real-valued function $\tau(t)$. Now notice that the above restrictions on G allows an arbitrariness in G of the form $G \to \mathrm{e}^{\mathrm{i}\gamma(t)\sigma_3}G$ for a real-valued function $\gamma(t)$. If we require that $\gamma(t)$ satisfies

$$\frac{\partial\gamma}{\partial t}(t) = \tau(t) - \rho^2$$

then G can be modified so that for the new G the second term on the right-hand side of (33) is $-\rho^2\sigma_3$. It implies that $M^G(\lambda)$ is exactly the $\mathrm{i}\{\partial_x^{-1}\partial_y U^2 - \rho^2 + U_y\}\sigma_3$ and hence ϕ satisfies the (2 + 1) NLS$^-$. This completes the proof of the gauge equivalence between the (2 + 1)-dimensional NLS$^-$ and the (2 + 1)-dimensional M-HF model.

3. Gauge equivalence between the $\mathrm{DS}^-_{\mathrm{II}}$ and the Minkowski IE

For the NLS$^-$, the different way from the above section in yielding (2 + 1)-dimensional integrable model is to replace the spectral parameter by a differential operator, such as ∂_y. In this process, one obtains the $\mathrm{DS}^-_{\mathrm{II}}$ (5), (6) with $\kappa = -1$. The $\mathrm{DS}^-_{\mathrm{II}}$ permits a Lax pair as follows:

$$P_1 = \mathrm{i}\partial_y - \sigma_3\partial_x + Q \tag{34}$$

$$P_2 = \partial_t + 2\mathrm{i}\sigma_3\partial_x^2 - 2\mathrm{i}U\partial_x + C \tag{35}$$

with

$$Q = \begin{pmatrix} 0 & q \\ -\bar{q} & 0 \end{pmatrix} \qquad C = \begin{pmatrix} \mathrm{i}a & -\mathrm{i}q_x + q_y \\ \mathrm{i}\bar{q}_x + \bar{q}_y & \mathrm{i}b \end{pmatrix} \quad b = -\bar{a} \tag{36}$$

and

$$-(b+a)_y + \mathrm{i}(b-a)_x = 2\mathrm{i}(q\bar{q})_x \tag{37}$$

$$-(b-a)_y + \mathrm{i}(b+a)_x = 2(q\bar{q})_y. \tag{38}$$

Now for the system of the Schrödinger flow of maps into the hyperbolic 2-space H^2 (i.e. [3, equation (5)]) or in other words, the M-HF model, to our surprise we must use the rejected form of Lax pair (7) displayed in [3] to obtain the following (2 + 1)-dimensional integrable equation (which we call the Minkowski IE) in the same way as the DS:

$$\boldsymbol{S}_t + \boldsymbol{S}\dot{\times}(\boldsymbol{S}_{xx} - \boldsymbol{S}_{yy}) + \phi_x\boldsymbol{S}_y + \phi_y\boldsymbol{S}_x = 0 \tag{39}$$

$$\phi_{xx} + \phi_{yy} + 2\boldsymbol{S}(\boldsymbol{S}_x\dot{\times}\boldsymbol{S}_y) = 0 \tag{40}$$

where $\boldsymbol{S} = (s_1(t,x,y), s_2(t,x,y), s_3(t,x,y)) \in R^{2+1}$ with $|\boldsymbol{S}|^2 = -1$ and $s_3 > 0$. The Lax pair of (39), (40) is

$$L_1 = \mathrm{i}\partial_y + S\partial_x \tag{41}$$

$$L_2 = \partial_t - 2\mathrm{i}S\partial_x^2 - (\mathrm{i}S_x - S_yS - \mathrm{i}\phi S - \phi_y)\partial_x \tag{42}$$

with

$$S = \begin{pmatrix} s_3 & -s \\ \bar{s} & -s_3 \end{pmatrix} \qquad s = s_1 + \mathrm{i}s_2.$$

Therefore, equations (39), (40) and the $\mathrm{DS}_{\mathrm{II}}^-$ can be represented as $[L_1, L_2] = 0$ and $[P_1, P_2] = 0$, respectively.

In this section, we show that the Minkowski IE (39), (40) is gauge equivalent to the $\mathrm{DS}_{\mathrm{II}}^-$. If we can find a gauge transformation T such that

$$P_1 = TL_1T^{-1} \qquad P_2 = TL_2T^{-1} \tag{43}$$

then one obtains $[P_1, P_2] = T[L_1, L_2]T^{-1}$, which indicates the gauge equivalence from (39), (40) to $\mathrm{DS}_{\mathrm{II}}^-$. By comparing the coefficients of ∂_x^j on both sides of (43), one finds that such a matrix T must satisfy

$$-\sigma_3 = TST^{-1} \tag{44}$$

$$Q = -\mathrm{i}T_yT^{-1} + \sigma_3T_xT^{-1} \tag{45}$$

$$2\mathrm{i}Q = 4\mathrm{i}\sigma_3T_xT^{-1} + T(\mathrm{i}S_x - S_yS - \mathrm{i}\phi_xS - \phi_y)T^{-1} \tag{46}$$

$$C = -T_tT^{-1} - 2\mathrm{i}\sigma_3T_{xx}T^{-1} + 2\mathrm{i}QT_xT^{-1}. \tag{47}$$

Firstly, suppose that the pair (S, ϕ) is a solution to (39), (40) and L_1, L_2 its corresponding Lax pair, we construct such a gauge transformation T. By solving (44), we see that the general form of T is

$$T = \mathrm{diag}(\lambda, \bar{\lambda})(S - \sigma_3) \tag{48}$$

where λ is temporally arbitrary. Substituting (48) into (45), by requiring that the right-hand side of (45) be off-diagonal, then the constraint for λ reads

$$\frac{\lambda_x}{\lambda} - \frac{s_{3x}}{2} + \frac{\bar{s}s_x}{2(s_3-1)} - \mathrm{i}\left(\frac{\lambda_y}{\lambda} - \frac{s_{3y}}{2} + \frac{\bar{s}s_y}{2(s_3-1)}\right) = 0 \tag{49}$$

and q is then given exactly in term of S and λ in the following way:

$$q = \frac{\lambda}{\bar{\lambda}}\left[\left(\frac{ss_{3x}}{2(s_3-1)} - \frac{s_x}{2}\right) - \mathrm{i}\left(\frac{ss_{3x}}{2(s_3-1)} - \frac{s_x}{2}\right)\right] \tag{50}$$

and, by using the identity $s_{1x}s_{2y} - s_{2x}s_{1y} = s_3\boldsymbol{S}\cdot(\boldsymbol{S}_x \dot{\times} \boldsymbol{S}_y)$, we obtain

$$q\bar{q} = \tfrac{1}{4}\left[|\boldsymbol{S}_x|^2 + |\boldsymbol{S}_y|^2 + 2\boldsymbol{S}\cdot(\boldsymbol{S}_x \dot{\times} \boldsymbol{S}_y)\right]. \tag{51}$$

Substituting T and q, given by (50), into (46) which give rise to an additional constraint on λ:

$$4\mathrm{i}\left(\frac{\lambda_x}{\lambda} - \frac{s_{3x}}{2} + \frac{\bar{s}s_x}{2(s_3-1)}\right) + \mathrm{i}\phi_x - \phi_y = 0. \tag{52}$$

Notice that equations (49) and (52) give rise to

$$4\left(\frac{\lambda_y}{\lambda} - \frac{s_{3y}}{2} + \frac{\bar{s}s_y}{2(s_3-1)}\right) - \mathrm{i}\phi_x + \phi_y = 0. \tag{53}$$

It is a direct calculation to verify that equation (12) (or equation (40)) plays the role of the compatibility condition for (52) and (53); in the calculation we have to use the identity $s_{1x}s_{2y} - s_{2x}s_{1y} = s_3\boldsymbol{S}\cdot(\boldsymbol{S}_x\dot{\times}\boldsymbol{S}_y)$.

For equation (47) we would like to verify that it is also satisfied through a direct proof, but not a lengthy calculation as done in [9]. Let us choose a $SU(1,1)$-valued matrix solution $\tilde{T}$ to

$$P_1\tilde{T} = 0 \qquad P_2\tilde{T} = 0 \tag{54}$$

with q given in (50) and the C given in (36) and (37), (38) which depends obviously on q. If we set $S = -\tilde{T}^{-1}\sigma_3\tilde{T}$ and represent the $\tilde{T}$ as $\tilde{T} = \mathrm{diag}(\tilde{\lambda}, \bar{\tilde{\lambda}})(S - \sigma_3)$, then from the equation $0 = (P_1\tilde{T})\tilde{T}^{-1} = \mathrm{i}\tilde{T}_y\tilde{T}^{-1} - \sigma_3\tilde{T}_x\tilde{T}^{-1} + Q$ (i.e. equation (45)) we obtain the result that the $\tilde{\lambda}$ also satisfies equation (49) if we insert $\tilde{\lambda}$ instead of λ. Furthermore, substituting $S = -\tilde{T}^{-1}\sigma_3\tilde{T}$ into (46) and using (45), we find

$$\mathrm{i}(\sigma_3\tilde{T}_x\tilde{T}^{-1} + \tilde{T}_x\tilde{T}^{-1}\sigma_3) - \tilde{T}_y\tilde{T}^{-1} - \sigma_3\tilde{T}_y\tilde{T}^{-1}\sigma_3 + \mathrm{i}\phi_x\sigma_3 - \phi_y = 0. \tag{55}$$

Hence

$$\phi_x = \tfrac{1}{2}\mathrm{i}\,\mathrm{tr}(\mathrm{i}\tilde{T}_x\tilde{T}^{-1} + \mathrm{i}\sigma_3\tilde{T}_x\tilde{T}^{-1}\sigma_3 - \sigma_3\tilde{T}_y\tilde{T}^{-1} - \tilde{T}_y\tilde{T}^{-1}\sigma_3) \tag{56}$$

$$\phi_y = \tfrac{1}{2}\mathrm{i}\,\mathrm{tr}(\sigma_3\tilde{T}_x\tilde{T}^{-1} + \tilde{T}_x\tilde{T}^{-1}\sigma_3 + \mathrm{i}\tilde{T}_y\tilde{T}^{-1} + \mathrm{i}\sigma_3\tilde{T}_y\tilde{T}^{-1}\sigma_3). \tag{57}$$

One can easily check that the compatibility condition of (56) and (57) is automatically satisfied (see [9, p L476]). Therefore, from (56) and (57) we have

$$\phi_{xx} + \phi_{yy} = -\tfrac{1}{2}\mathrm{i}\,\mathrm{tr}\big(\sigma_3[\tilde{T}_x\tilde{T}^{-1}, \tilde{T}_y\tilde{T}^{-1}] + [\tilde{T}_x\tilde{T}^{-1}, \tilde{T}_y\tilde{T}^{-1}]\sigma_3\big) \tag{58}$$

or equivalently

$$\phi_{xx} + \phi_{yy} = -\mathrm{i}\,\mathrm{tr}\left(S[S_x, S_y]\right). \tag{59}$$

Equation (59) is the same as (40) or (12), but in a different form. This shows that the above $\tilde{T}$ satisfies (46) or, in other words, $\tilde{\lambda}$ satisfies (49) and the additional constraint (52). Thus λ and $\tilde{\lambda}$ satisfy the following equations which can be obtained directly from (49) and (52):

$$\frac{\lambda_x}{\lambda} = \frac{\tilde{\lambda}_x}{\tilde{\lambda}} = \left(\frac{1}{4}(-\phi_x - \mathrm{i}\phi_y) + \frac{s_{3x}}{2} - \frac{s_x\bar{s}}{2s_3 - 2}\right) \tag{60}$$

$$\frac{\lambda_y}{\lambda} = \frac{\tilde{\lambda}_y}{\tilde{\lambda}} = \left(\frac{1}{4}(\mathrm{i}\phi_x - \phi_y) + \frac{s_{3y}}{2} - \frac{s_y\bar{s}}{2s_3 - 2}\right). \tag{61}$$

Hence we obtain $\lambda = p(t)\tilde{\lambda}$ for some non-vanishing complex function p. Furthermore, by using $S = -T^{-1}\sigma_3 T = -\tilde{T}^{-1}\sigma_3\tilde{T}$, we see that p is in fact a real function, i.e. $T = \mathrm{diag}(p(t), p(t))\tilde{T}$. From this relation and the equation $P_2\tilde{T} = 0$, we find that the off-diagonal part of $-T_tT^{-1} - 2\mathrm{i}\sigma_3T_{xx}T^{-1} + 2\mathrm{i}UT_xT^{-1}$ is just the same part of C given by (36) with q given in (50) and a, b determined by (37), (38) and diagonal part of $-T_tT^{-1} - 2\mathrm{i}\sigma_3T_{xx}T^{-1} + 2\mathrm{i}UT_xT^{-1}$ is

$$\mathrm{diag}\left(\frac{p'(t)}{p(t)}, \frac{p'(t)}{p(t)}\right) + C^{(\mathrm{diag})}$$

and therefore it satisfies the same equations (37), (38). This shows that T satisfies (47) and hence that q and its associated a, b constructed from the solution (S, ϕ) to (39), (40) satisfies the $\mathrm{DS}_{\mathrm{II}}^-$.

Next we shall prove that the above transformation from (39), (40) to the $\mathrm{DS}_{\mathrm{II}}^-$ is in fact reversible. Suppose q and its associated a, b are solutions to the $\mathrm{DS}_{\mathrm{II}}^-$. Let us choose the $SU(1,1)$-valued matrix solution T to

$$P_1T = 0 \qquad P_2T = 0 \tag{62}$$

and define

$$S = -T^{-1}\sigma_3 T \tag{63}$$

i.e. T satisfies (44), (45) and (47) with the given q. To determine ϕ such that (46) and (12) are valid, we substitute (63) into (46). Then with the same arguments as above for $\tilde{T}$, we get that the ϕ given by (56) and (57) satisfies (58) or (59). Therefore, we have proved that the matrix T satisfing (62) gives rise to the gauge transformation from $\mathrm{DS}_{\mathrm{II}}^{-}$ to (39), (40) or (11), (12).

4. Conclusion and remarks

In this paper, we have proved that the two analogous models of the NLS^{-} in 2 + 1 dimensions are gauge equivalent to those of the Schrödinger flow of maps into hyperbolic 2-space H^2 (i.e. the M-HF model) in 2 + 1 dimensions, respectively. These are generalized versions of the result obtained from [3] in two different ways and demonstrate the deep relations between those pairs of two $(2+1)$-dimensional integrable systems. Combining the present results with those of [9] and [10], we have a complete understanding of the gauge equivalence of the two analogous models of the NLS (1) for $\kappa = 1$ or -1 and those of the Euclidean or Minkowski HF model in 2 + 1 dimensions. The results indicate that the equations of the analogues of the NLS^{+} and NLS^{-} in 2 + 1 dimensions can be explained geometrically as being of 'elliptic type' and 'hyperbolic type', respectively, since the targets of their corresponding Schödinger flow in 1 + 1 dimensions are a Euclidean 2-sphere and a hyperbolic 2-space. In a physical sense, we again see from the above results that the light-soliton solutions and dark-soliton solutions of the NLS in 1 + 1 or 2 + 1 dimensions should have the character of localized and non-localized solutions, respectively. We believe that these results are interesting and important in the integrable theory of the NLS and the HF models both in 1 + 1 and 2 + 1 dimensions. By the way, we also find the dynamical meanings of the modifying function u in the system of the $(2+1)$-dimensional M-HF model (17) (or also in the $(2+1)$-dimensional HF model) explicitly. That is, if we let F be a solution to its Lax pair and set $F_y = VF$ for some matrix V, then $\mathrm{diag}(-u, u)$ is simply the diagonal part of the matrix $V|_{\lambda=0}$. This gives an interesting dynamical explanation for u.

However, many questions remain open and deserve further investigation in this respect. Examples are: finding physical applications of the results obtained in this paper, whether one can find a Schrödinger-like equation which is gauge equivalent to the generalized Landau–Lifshitz equation [11] (or see [1, 12]) and, in 2 + 1 dimensions, whether the DS_{I} for $\kappa = 1$ or -1 (see, for example [4]) exists with such a gauge equivalent structure.

Acknowledgments

The author would like to express his thanks to Dr Z X Zhou for his interest and helpful discussions in this work. He also very grateful to Professors C H Gu and H S Hu for their encouragement. This work is partially supported by STDF of Shanghai, China, NNSF (19631130) of China and CNPq of Brazil.

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